Deptt- MATHEMATICS

College- SOGHRA COLLEGE,BIHAR SHARIF

Part- BSc PART 2

Solved Examples

1:- Linear differential Equations with constant coefficients:-

ex(1) Solve $\frac{d^2y}{dx^2} - 9y = 6e^{3x} + xe^{3x}$

Soln:- Auxiliary equation is $D^2 - 9 = 0 \Rightarrow D = \pm 3$

$\therefore$ C.F. $= c_1 e^{3x} + c_2 e^{-3x}$

P.I. $= \frac{1}{D^2-9} e^{3x}(x+6) = e^{3x} \frac{1}{(D+3)^2 - 9}(x+6)$

$= e^{3x} \cdot \frac{1}{D^2+6D}(x+6) = e^{3x} \frac{1}{6D}\left(1 + \frac{1}{6}D\right)^{-1}(x+6)$

$= e^{3x} \cdot \frac{1}{6D}\left(1 - \frac{1}{6}D\right)(x+6)$

$= e^{3x} \frac{1}{6D}\left(x + 6 - \frac{1}{6}\right) = \frac{1}{36} e^{3x}(3x^2 + 35x)$

$\therefore$ complete solution is $y = c_1 e^{3x} + c_2 e^{-3x} + \frac{1}{36} e^{3x}(3x^2 + 35x)$

ex(2) Solve $\frac{d^3y}{dx^3} - 13\frac{dy}{dx} - 12y = 0$

Soln:- A.E. is $D^3 - 13D - 12 = 0$

$\Rightarrow (D+1)(D+3)(D-4) = 0 \Rightarrow D = -1, -3, 4$

$\therefore$ complete solution is $y = c_1 e^{-x} + c_2 e^{-3x} + c_4 e^{4x}$

ex(3) Solve $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0$

Soln:- A.E is $D^4 - D^3 - 9D^2 - 11D - 4 = 0$

$\Rightarrow (D+1)^3(D-4) = 0 \Rightarrow D = -1, -1, -1, 4$

$\therefore$ General solution is $y = (c_1 + c_2 x + c_3 x^2)e^{-x} + c_4 e^{4x}$

ex 4 solve $(D^3 - 2D^2 - 4D + 8)y = 0$

Soln:- A.E. is $D^3 - 2D^2 - 4D + 8 = 0$

$\Rightarrow (D+2)(D-2)^2 = 0 \Rightarrow D = -2, 2, 2$

$\therefore$ Solution is $y = (c_1 + c_2 x)e^{2x} + c_3 e^{-2x}$

ex 5 $(D^4 + 5D^2 + 6)y = 0$

Soln:- A.E. is $D^4 + 5D^2 + 6 = 0$

$\Rightarrow (D^2+2)(D^2+3) = 0 \Rightarrow D = \pm\sqrt{2}\,i, \pm\sqrt{3}\,i$

$\therefore$ complete solution is $y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x + c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x$

ex 6 solve $(D^4 + D^3 + D + 1)y = 0$

Soln:- A.E. is $D^4 - D^3 + D + 1 = 0$

$\Rightarrow (D^3 - 1)(D + 1) = 0$

$\Rightarrow (D-1)^2(D^2 + D + 1) = 0$

$\Rightarrow D = -1, -1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\therefore$ complete solution is

$y = (c_1 + c_2 x)e^{-x} + e^{-x/2}\left(c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x\right)$

ex 7 solve $(D^4 - 4D^3 + 8D^2 - 8D + 4)y = 0$

Soln:- A.E. is $D^4 - 4D^3 + 8D^2 - 8D + 4 = 0$

$\Rightarrow (D-1)^2(D+2)^2 = 0 \Rightarrow D = 1, 1, -2, -2$

$\therefore$ Solution is $y = (c_1 + c_2 x)e^{x} + (c_3 + c_4 x)e^{-2x}$

ex(8) Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$

Soln:- A.E. is $D^2 + D + 1 = 0 \Rightarrow D = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\therefore$ C.F. $= e^{-\frac{1}{2}x}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right)$

$$P.I. = \frac{1}{D^2+D+1}\sin 2x = \frac{1}{-4+D+1}\sin 2x$$

$$= \frac{1}{D-3}\sin 2x = \frac{D+3}{D^2-9}\sin 2x$$

$$= -\frac{1}{13}(2\cos 2x + 3\sin 2x)$$

ex(9) Solve $(D^2 - 5D + 6)y = \sin 3x$

Soln:- A.E. is $D^2 - 5D + 6 = 0 \Rightarrow D = 2, 3$

$\therefore$ C.F. $= c_1 e^{2x} + c_2 e^{3x}$

Now $P.I. = \frac{1}{D^2 - 5D + 6}\sin 3x = \frac{1}{-3^2 - 5D + 6}\sin 3x$

$$= \frac{1}{-(5D+3)}\sin 3x = -\frac{(5D-3)}{25D^2-9}\sin 3x$$

$$= \frac{-1}{25(-3^2)-9}(5 \cdot 3\cos 3x - 3\sin 3x)$$

$$= \frac{1}{234}(15\cos 3x - 3\sin 3x)$$

$$= \frac{1}{78}(5\cos 3x - \sin 3x)$$

$\therefore$ complete solution is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{78}(5\cos 3x - \sin 3x)$$

ex (10) Solve $(D^2-3D+2)y = \cos 3x$

Soln:- A.E. is $D^2-3D+2=0 \Rightarrow D = 1, 2$

$\therefore$ C.F. $= c_1 e^{x} + c_2 e^{2x}$

Now P.I $= \dfrac{1}{D^2-3D+2}\cos 3x = \dfrac{1}{-9-3D+2}\cos 3x$

$= \dfrac{1}{-(3D+7)}\cos 3x = -\dfrac{(3D-7)}{9D^2-49}\cos 3x$

$= \dfrac{-(-3\cdot 3\sin 3x - 7\cos 3x)}{9(-9)-49}$

$= -\dfrac{1}{130}(9\sin 3x + 7\cos 3x)$

$\therefore$ complete solution is

$y = c_1 e^{x} + c_2 e^{2x} - \dfrac{1}{130}(9\sin 3x + 7\cos 3x)$

ex (11) Solve $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^x$

Soln:- A.E. is $D^2-3D+2=0 \Rightarrow D = 1, 2$

$\therefore$ C.F. $= c_1 e^{x} + c_2 e^{2x}$

P.I $= \dfrac{1}{D^2-3D+2}e^x = x\dfrac{e^x}{2D-3} = \dfrac{xe^x}{2\cdot 1-3} = -xe^x$

$\therefore$ complete solution is $y = c_1 e^{x} + c_2 e^{2x} - xe^x$

ex (11) Solve $(D^2+4D+3)y = e^{-3x}$

Soln:- A.E. is $D^2+4D+3=0 \Rightarrow D = -1, -3$

$\therefore$ C.F. $= c_1 e^{-x} + c_2 e^{-3x}$

P.I $= \dfrac{1}{D^2+4D+3}e^{-3x} = \dfrac{xe^{-3x}}{2D+4} = \dfrac{xe^{-3x}}{2(-3)+4} = -\dfrac{1}{2}xe^{-3x}$

$\therefore$ complete solution is $y = c_1 e^{-x} + c_2 e^{-3x} - \dfrac{1}{2}xe^{-3x}$

ex (12) Solve $(D^3+2D^2+D)y = e^{2x} + x^2 + x$

Soln:- A.E. $= D(D^2+2D+1) = 0 \Rightarrow D = 0, -1, -1$

$\therefore$ C.F. $= C_1 + (C_2 + C_3x)e^{-x}$

$$P.I. = \frac{1}{D(D+1)^2}(e^{2x} + x^2 + x)$$

$$= \frac{1}{D(D+1)^2}e^{2x} + \frac{1}{D(D+1)^2}(x^2+x)$$

$$= \frac{e^{2x}}{2(2+1)^2} + \frac{1}{D}(1+D)^{-2}(x^2+x)$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}(1-2D+3D^2-\cdots)(x^2+x)$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}(x^2+x-4x-2+6)$$

$$= \frac{e^{2x}}{18} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x$$

$\therefore$ complete solution is

$$y = C_1 + (C_2 + C_3x)e^{-x} + \frac{e^{2x}}{18} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x$$

ex (13) Solve $(D^2-9)y = 6e^{3x} + xe^{3x}$

Soln:- $D^2 - 9 = 0 \Rightarrow D = \pm 3$

C.F. $= C_1e^{3x} + C_2e^{-3x}$

$$P.I. = \frac{1}{D^2-9}e^{3x}(6+x) = e^{3x}\cdot\frac{1}{(D+3)^2-9}(6+x)$$

$$= e^{3x}\cdot\frac{1}{D^2+6D}(6+x)$$

$$= e^{3x}\cdot\frac{1}{6D}\left(1+\frac{1}{6}D\right)^{-1}(6+x)$$

$$= e^{3x}\cdot\frac{1}{6D}\left(1-\frac{1}{6}D+\cdots\right)(6+x)$$

$$= e^{3x}\cdot\frac{1}{6D}\left(6+x-\frac{1}{6}\right) = \frac{1}{36}e^{3x}(35x+3x^2)$$

$\therefore$ complete solution is $y = C_1e^{3x} + C_2e^{-3x} + \frac{1}{36}e^{3x}(35x+3x^2)$

ex (14) Solve $\frac{d^2y}{dx^2} + 4y = x\sin x$

Soln:- A.E:- $D^2+4=0 \Rightarrow D = \pm 2i$

$C.F. = C_1\cos 2x + C_2\sin 2x$

$P.I. = \frac{1}{D^2+4} x\sin x = I.P.\ of\ \frac{1}{D^2+4} x e^{ix}$

$= I.P.\ of\ e^{ix}.\frac{1}{(D+i)^2+4} x$

$= I.P.\ of\ e^{ix}.\frac{1}{D^2+2Di+3} x$

$= I.P.\ of\ e^{ix}.\frac{1}{3} e^{ix}\left(1+\frac{2}{3}Di+\frac{1}{3}D^2\right)^{-1} x$

$= I.P.\ of\ \frac{1}{3}e^{ix}\left(1-\frac{2}{3}Di\right) x$

$= I.P.\ of\ \frac{1}{3}(\cos x + i\sin x)\left(x-\frac{2}{3}i\right)$

$= \frac{1}{9}(3x\sin x - 2\cos x)$

$\therefore$ Complete solution is $y = C_1\cos 2x + C_2\sin 2x + \frac{1}{9}(3x\sin x - 2\cos x)$

ex (15) Solve $\frac{d^2y}{dx^2} - y = x^2\cos x$

Soln:- A.E:- $D^2-1=0 \Rightarrow D = \pm 1$

$C.F. = C_1e^{x} + C_2e^{-x}$

$P.I. = \frac{1}{D^2-1} x^2\cos x = R.P.\ of\ \frac{1}{D^2-1} x^2 e^{ix}$

$= R.P.\ of\ e^{ix}.\frac{1}{(D+i)^2-1} x^2$

$= R.P.\ of\ e^{ix}.\frac{1}{D^2+2Di-2} x^2 = R.P.\ of\ e^{ix}\left(-\frac{1}{2}\right)\left(1-iD-\frac{1}{2}D^2\right)^{-1} x^2$

$= R.P.\ of\ e^{ix}\left(-\frac{1}{2}\right)\left(1+iD-\frac{1}{2}D^2\right) x^2$

$= R.P.\ of\ -\frac{1}{2}(\cos x + i\sin x)(x^2+2ix-1)$

$P.I = -\frac{1}{2}\{(x^2-1)\cos x - 2x\sin x\}$

$\therefore$ complete solution is $y = C.F. + P.I$

PART-II (PAPER-III)

ABSTRACT ALGEBRA

CHAPTER:- Sub-Groups

① Definition:- Let G be a group and H a subset of G. Then H is called to be a subgroup of G if H is a group under the group operation of G. It is sometimes written as $H \leq G$.

Necessary and Sufficient Condition for a Subgroup

Theorem:- Let G be a group and H a non-empty subset of G. Then H is a subgroup of G if and only if (i) for all $a \in H$, $b \in H \Rightarrow ab \in H$ i.e. H is closed under the given operation and (ii) for all $a \in H$, the inverse of a i.e. $a^{-1} \in H$.

Necessary Condition:- H is a given to be a group under the group operation of G. Now since H is a group, therefore for all $a, b \in H$, we have $ab \in H$ and the first condition is satisfied. Also, for all $a \in H$, we have $a^{-1} \in H$ and the second condition is satisfied. Thus if H is a subgroup of G, the condition (i) and (ii) are satisfied and the first part of the theorem is proved.

Sufficient Condition:- We must show that H is a group under the operation of G. We shall show that with the given two conditions, H satisfies all the four postulates of a group.

(i) We are given that for all $a, b \in H$, $ab \in H$ and hence the first postulate is satisfied.

(ii) H is associative because H is a subset of G which is associative.

(iii) If $x \in H$, then because of condition (ii), $x^{-1} \in H$ and the fourth postulate is satisfied.

(iv) Again we have by (i) $x \cdot x^{-1} \in H$, i.e. $e \in H$. Hence the identity element of H is necessarily e.

Thus we see that H satisfies all the four postulates of a group and hence it is a subgroup.

(1)

(3)

(2) Theorem:- Prove that a necessary and sufficient condition for a non-empty subset H of a group G be a subgroup is that
$a \in H, b \in H \Rightarrow ab^{-1} \in H$ where b^{-1} is the inverse of b in G.

Proof:- ~~Necess~~ The condition is sufficient:- We shall first prove that the condition sufficient. We shall prove that if $a \in H$, $b \in H \Rightarrow ab^{-1} \in H$, then H is a group. In this connection we should remember that if $a \in H$, then $a \in G$ also, since H is a subset of G. Given that $\{a \in H, b \in H\} \Rightarrow ab^{-1} \in H$.

Existence of identity:- In this relation taking $b = a$, we get

$\{a \in H, b \in H\} \Rightarrow a a^{-1} \in H$ where a^{-1} is the inverse of a in G.

$\Rightarrow e \in H$, where e is the identity of G.

Hence e is an identity of H also.

This postulate 3 is satisfied.

Existence of Inverse:- Again, let $a \in H$. Then $\{e \in H, a \in H\}$

$\Rightarrow e.a^{-1} \in H$ That is, $a^{-1} \in H$.

Thus if $a \in H$, then its inverse $a^{-1} \in H$

Thus postulate 4 is satisfied.

Closure Property:- Now $\{a \in H, b^{-1} \in H\} \Rightarrow a(b^{-1})^{-1} \in H$

$\Rightarrow ab \in H$. Hence $\{a \in H, b \in H\} \Rightarrow ab \in H$.

Thus Postulate 1 is satisfied

Associatively:- The binary operation in G is associative and since it is a subset of G, it must be associative in H also. Thus Postulate 2 is satisfied.

Hence the set H forms a group

The Condition is necessary:- We shall prove that if H is a group and $a, b \in H$ then $ab^{-1} \in H$.

Given that H is a group and $b \in H$, then its inverse $b^{-1} \in H$ also. Therefore according to the first postulate $ab^{-1} \in H$ since H is a group. Thus the above condition has been proved to be necessary.

Theorem:- If H_1 and H_2 be two Subgroups of a group G, then $H_1 \cap H_2$ is also a subgroup of G. OR

Prove that intersection of two subgroup is a subgroup.

Proof:- Let x_1 and x_2 both $\in H_1 \cap H_2$

Then $x_1, x_2 \in H_1$ and also $x_1, x_2 \in H_2$

Therefore, $x_1 x_2 \in H_1$ and $x_1 x_2 \in H_2$

Hence $x_1 x_2 \in H_1 \cap H_2$

Again $x_1 \in H_1$ and $x_1 \in H_2$

$\therefore$ $x_1^{-1} \in H_1$ and $x_1^{-1} \in H_2$

Hence $x_1^{-1} \in H_1 \cap H_2$

~~[illegible]~~

Thus $H_1 \cap H_2$ is a subgroup of G.

(1) **Cyclic Group:-** Let (G, o) be a group. If there exists $a \in G$ such that $G = \{a^i / i \in I\}$, where I is the set of all integers, then G is called a cyclic group and a is called the general of the cyclic group.

Hence the group G is said to be cyclic if it is capable of being generated by some element.

There can be more than one generators of a cyclic group.

Illustration:- $(Z, +)$ is a cyclic group generated by 1, where Z is the set of integers.

Q Prove that every cyclic group is an Abelian group

Ans:- Let G be a cyclic group generated by a.
let x, y be any two elements $\in G$.
Then there exist integers r and s such that $x = a^r$ and $y = a^s$
Now, $xy = a^r \cdot a^s = a^{r+s}$ and $y \cdot x = a^s \cdot a^r = a^{s+r} = a^{r+s}$

$\therefore$ $xy = yx$ for all $x, y \in G$

Hence G is an Abelian Group

Q:- Prove that if a is generator of a cyclic group G, then a^{-1} is also a generator of G.

Ans:- Let G be a cyclic group generated by a
Let a^r be any element of G where r is some integer
We can write a^r as $(a^{-1})^{-r}$. Since $-r$ is also some integer, therefore G is also generated by a^{-1}
Hence a^{-1} is also a generator of G.

Q:- Prove that if a finite group of order n contains an element of order n, the group must be cyclic.

Ans- Let G be a finite group of order n.
Let $a \in G$ and let n be the order of a i.e. n is the least positive integer such that $a^n = e$, the idendity element of G. It is to prove that G is cyclic.
let $H = \langle a \rangle = \{a, a^2, a^3, \dots a^n = e\}$
Then H is a subgroup of G and $O(H) = n$
Since $H \subseteq G$ and $O(H) = O(G)$ therefore $G = H = \langle a \rangle$
Hence G is cyclic.

Q:- Prove that every subgroup H of a cyclic group G is also a cyclic group.

Ans: Suppose that the cyclic group G is generated by a and let H be a subgroup of G.

Then every member of H will be evidently some integral powers of a, positive or negative.
Let m be the smallest positive integer such that $a^m \in H$
We shall show that H is a cyclic subgroup generated by a^m.
Let a^k be any element of H
Obviously $k > m$.
By the division algorithm, we may write $k = qm + r$,
where $0 \leq r < m$ Hence $a^k = a^{qm+r} = (a^m)^q . a^r$

$\Rightarrow a^r = a^k . (a^m)^{-q}$

Since $a^m \in H$ and $a^k \in H$. this equation implies that $a^r \in H$

But m is the smallest positive integer such that $a^m \in H$.

Since $r < m$. We must have $r = 0$.

Therefore $k = qm$ and hence every element a^k of H is of the form $(a^m)^q$ for some integer K.
This show that H is a cyclic group generated by a^m
i.e. a^m is generator of H.

Q:- Prove that if H is a subgroup of the cyclic group G, then the order of G is a multiple of the order of H.

ANS:- Suppose the order of the cyclic group generated by a is n. Then $G = \{a, a^2, a^3, \ldots\ldots a^{n-1}, a^n = e\}$
Let H be a proper subgroup of G.
Since the order of the cyclic group is n, therefore the order of a is also $n (>0)$ so that $a^n = e$
$\therefore a^n \in H$, since $e \in H$
Every number of H will evidently be some integral powers of a, positive or negative

Let m be the smallest of those positive integers such that $a^m \in H$.
We know that H is a cyclic subgroup generated by a^m. Let the order of a^m, the generator of H be q.
Then $(a^m)^q = e = a^n \Rightarrow a^{mq} = a^n \Rightarrow mq = n$
$\therefore$ H consists of q distinct elements.

$$H = \{a^m, a^{2m}, \ldots a^{qm}\}$$

Hence the subgroup H of a finite cyclic group G of order n is a finite cyclic group of order $q = \frac{n}{m}$. From the question, we have $m = \frac{n}{q}$ i.e. the order of H devides the order of G.

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Q:- State and Prove Uniqueness of Identity or.
The identity element in a group is Unique.

Ans:- An element $e \in G$ is said to be an identity for the operation in G if $ae = ea = a \ \forall \ a \in G$.
We want to prove that e is unique. If possible, let e and e' be two identity elements both belong to G.
Then e is an identity element $\Rightarrow ee' = e'$
Again e' is an identity element $\Rightarrow ee' = e$
So we find that $e' = e$. Hence e is Unique.

Q:- To state and Prove Uniqueness of Inverse. OR
The inverse of an element in a group is Unique

Ans:- An element b is said to be inverse of an element a if $ab = ba = e \in G$, for $a, b \in G$.
Let the inverse be not unique
Let b and $c \in G$ be two distinct inverse of a

Then we have
$ab = ba = e$ and $ac = ca = e$
Now $(ba)c = ce = c$, $a, b, c \in G$
But $(ba)c = b(ac)$, by associative property.
$\therefore c = b$
Hence the inverse is unique.

Q:- Prove that the set to cube roots of unity forms an abelian group with respect to multiplication.

Ans:- The set of cube root of unity is $\{1, \omega, \omega^2\}$
where ω is an imaginary cube root of unity

(i) Since $1 \cdot 1 = 1$, $1 \cdot \omega = \omega$, $1 \cdot \omega^2 = \omega^2$
$\omega \cdot 1 = \omega$, $\omega \cdot \omega = \omega^2$, $\omega \cdot \omega^2 = \omega^3 = 1$
$\omega^2 \cdot 1 = \omega^2$, $\omega^2 \cdot \omega = \omega^3 = 1$, $\omega^2 \cdot \omega^2 = \omega^4 = \omega$
Therefore the multiplication is a binary operation on the given set.

(ii) Since $(1 \cdot \omega) \cdot \omega^2 = 1 \cdot (\omega \cdot \omega^2)$ etc So the multiplication is associative.

(iii) Identity is 1, Since $1 \cdot \omega = \omega$, and $1 \cdot \omega^2 = \omega^2$, $1 \cdot 1 = 1$ ~~respectively~~ and it belong to the set.

(iv) The inverse of $1, \omega, \omega^2$ are respectively $1, \omega^2, \omega$ which belong to set.

(v) The commutative property is satisfied, Since $1 \cdot \omega = \omega \cdot 1$, $\omega \cdot \omega^2 = \omega^2 \cdot \omega$ etc.
Hence the given set is ~~of the~~ an abelian group under multiplication

Q Prove that the set $\{1, -1, i, -i\}$ is an abelian group under multiplication

Ans:- (i) $1\cdot(-1) = -1$, $1\cdot i = i$, $1(-i) = -i$, $i(-i) = 1$, $1\cdot 1 = 1$
So multiplication is a binary composition for the given set

(ii) Associative law is satisfied,
$(1\cdot i)(-i) = 1$, $\{i(-i)\} = i$
$(1\cdot i)(-1) = 1\{i(-1)\} = -i$ etc

(iii) Identity is 1 since $1\cdot(-1) = -1$
$1\cdot i = i$, $1\cdot(-i) = -i$ and 1 belong to the given set.

(iv) Inverse of all element in the set exist and belong to the set. Since $1\cdot 1 = 1$, $(-1)(-1) = 1$, $i(-i) = 1$, $(-i)i = 1$

(v) The commutative law is also satisfied as
$1(-1) = (-1)(1) \neq (-1)i = i(-1)$ etc
Hence the given set is an abelian group under multiplication

Q Prove that every homomorphic image of an Abelian group is Abelian.

Ans Let G be an Abelian group. Let f be a homomorphic mapping of G onto G'. Then G' is a homomorphic image of G.
It is to prove that G' is Abelian
Let a', b' be any two elements of G'
Then $f(a) = a'$ and $f(b) = b'$ for some $a, b \in G$.
we have $a'b' = f(a)f(b) = f(ab)$
$\because$ f is homomorphic mapping
$= f(ba)$ $\because$ G is Abelian
$= f(b)f(a) = b'a'$
Hence G' is Abelian ——x——

Q Prove that Cyclic groups of same order are isomorphic

Ans:- Let G and G' be two Cyclic groups of order n which are generated by a and b respectively.

Then $G = \{a, a^2, a^3, \dots a^n = e\}$ and $G' = \{b, b^2, b^3, \dots b^n = e'\}$

We shall show that the mapping defined by $f(a^r) = b^r$ is isomorphism.

For, $f(a^r \cdot a^2) = f(a^{r+2}) = b^{r+2} = b^r \cdot b^2 = f(a^r) \cdot f(a^2)$

$\therefore$ The groups are isomorphic.

—x—

Q:- Define Order of an element in a group.

Ans:- Definition:- Let (G, 0) be any group, then the least positive integer n iff its exists such that $a^n = e$ is Called order of $a \in G$.

If, however, there does not exist n for which $a^n = e$, then the order of G is infinite.

The identity of every group has order 1

The order of an element $a \in G$ is denoted by $O(a)$

Illustration:- let us consider the group

$\{a, a^2, a^3, a^4\}$ where $a^4 = e$

The order of $a = 4$ because 4 is the least Positive integer such that $a^4 = e$

The order of $a^2 = 2$ as 2 is the least positive integer such that $(a^2)^2 = a^4 = e$

The order of $a^3 = 4$ Since 4 is the least Positive integer such that $(a^3)^4 = a^{12} = (a^4)^3 = e$

Similarly the order of a^4 is 1.

Q Define Normal Sub-group with Example.

Ans. A sub-group M of a group G is called a Normal Sub-group or distinguished sub-group of G if for all $m \in M$ and $a \in G$. $\therefore$ $ama^{-1} \in M$.

Example ① If G is an abelian group, every subgroup M of G is normal in G. Infact for $m \in M$ and $a \in G$ we have $ama^{-1} = aa^{-1}m = em = m \in M$

② For any Group G the trivial group $\{e\}$ is normal in G, Similarly G is normal in G.

Q:- Define Simple group:

Ans:- A group G is called as Simple group if its only normal subgroups are $\{e\}$ and G.

Q:- Prove that every cyclic group of order n is isomorphic to the additive group Z_n of residue class modulo n

Ans:- Let $(G, .)$ be a cyclic group of order n with generator a, we define $f: Z_n \to G$ by

$$f([k]) = a^k \text{ for all } [k] \in Z_n$$

We know that $a^i = a^k$ iff $i \equiv k \pmod{n}$

Hence $a^i = a^k$ iff $[i] = [k]$

Hence f is well defined. Clearly f is one-to-one onto. Now, $f([i]+[k]) = f([i+k])$

$= a^{k+i} = a^i . a^k$

$= f[i] . f[k]$

Thus f is an isomorphism of the additive group Z_n onto G.

(1) State and Prove Lagrange's Theorem.

Ans: Statement :- The order of each sub-goup of a finite group is a divisor of the order of the group.
In other words if G is finite group of order n, the order of every sub-group H of G is a divisor of n.

Proof :- Let G be a group of infinite order n. Let H be a subgroup of G and let $O(H) = m$.

Suppose h_1, h_2, h_m are the m elements of H.

Let $a \in G$. Then Ha is a right Coset of H in G.
and we have $Ha = \{h_1a, h_2a \ldots h_ma\}$
Ha has m distinct members.

If h_i and h_j are two disjoint elements of H for which $h_ia = h_ja$. Then by right cancellation
$h_ia = h_ja \Rightarrow h_i = h_j$ which contradicts

Therefore each left coset of H in G has m distinct members. It has to be remembered that two left (right) Cosets are either equal or have no point in Common

Any two distinct left Cosets of H in G. are disjoint i.e. they have no element in common.
Since G is a finite group, the number of distinct left Cosets of H in G will be finite, say equal to k.
Also, the union of these k distinct left cosets of H in G
Thus if $Ha_1, Ha_2 \ldots Ha_k$ are the k distinct left cosets of H in G. then

$$G = Ha_1 \cup Ha_2 \cup Ha_3 \cup \ldots Ha_k \quad \text{——— (1)}$$

Two distinct left Cosets are mutually disjoint and each left coset has m members. Hence the total number of elements in the union of k disjoint left coset will be km and this will be equal to the number of elements in G.

Hence $n = km \Rightarrow \frac{n}{m} = k \Rightarrow m$ is a divisor of n.

$\Rightarrow o(H)$ is a divisor of $O(G)$

Hence the theorem.

Q Prove that $(ab)^2 = a^2b^2$ for all $a, b \in G$ which is a group is an abelian group

Ans- Let in the group G, $(ab)^2 = a^2b^2$

$\therefore (ab)(ab) = (aa)(bb)$

or, $a^{-1}\{(ab)(ab)\} = a^{-1}\{(aa)(bb)\}$

or, $(a^{-1}a)\{b(ab)\} = (a^{-1}a)\cdot\{a(bb)\}$ as associative law holds in a group

$\because a^{-1}a = e$

or, $e\{b(ab)\} = e\{a(bb)\}$

or, $b(ab) = a(bb)$, by the definition of identity element

or, $(ba)b = (ab)b$.

By right cancelation law $ba = ab$

i.e. G is an abelian

Next, let G be an abelian group. Then $ab = ba$ for all $a, b \in G$. Now, $(ab)^2 = (ab)(ab) = a(ba)b$ as associative law holds in a group

$= a(ab)b$ for $ab = ba$

$= (aa)(bb) = a^2b^2$ //

Q State and prove Fundamental theorem of homomorphism for group or State and prove the first Isomorphism theorem

Ans:- Statement:- (i) A quotient group G/M of a group G with respect to a normal sub-group M is a homomorphic image of G.

(ii) Conversely if a group G' is a homomorphic image of a group G then G' is isomorphic to a quotient group ~~the~~ in fact o the quotient group of G with respect to the kernel of the homomorphism

Proof:- (i) Let M be a normal subgroup of a group G and G/M is the corresponding quotient group, then we show that the map $h: G \to G/M$ given by $h(a) = aM$ for all $a \in G$, is homomorphism of G onto G/M

For $a, b \in G$, we have $h(ab) = abM = aM \cdot bM = h(a \cdot hb)$.
Hence h is a homomorphism, Moreover, any element of G/M is of the form aM for some $a \in G$ and hence aM is the image of a. Hence h is a homomorphism of G onto G/M. ~~Moreover, any element of G/M is of the form aM for some a∈G.~~
Thus G/M is a homomorphic image of G. The homomorphism h is called the natural homomorphism of G onto its quotient group G/M. We next claim that $\text{Ker } h = \{a \in G : h(a) = M\} = M$. Since $h(a) = aM$
$h(a) = M$ iff $aM = M$, that is iff $a \in M$. Thus $\text{Ker } h = M$.

Proof (ii) Let $f: G \to G'$ be a homomorphism of a group G onto a group G' and let $M = \text{Ker } f$. We show that M is a normal Sub-group of G and that f induces an isomorphism $f: G/M \to G'$
i.e. $G/M \simeq G'$

We know that M is a normal sub-group of G, Hence we may form G/M.

Now, let $\bar{f}: G/M \to G'$ be defined by $\bar{f}(aM) = f(a)$ for all $a \in G$. Firstly, we must show that $\bar{f}$ is well defined, that is, if $aM = bM$ then $f(a) = f(b)$. Now if $aM = bM$ then $a^{-1}b \in M$, and hence $f(a^{-1}b) = e'$, where e' is the identity of G'. Then $e' = f(a^{-1}b) = f(a^{-1}) \cdot f(b) = \{f(a)\}^{-1} f(b)$.

Hence $f(a) = f(b)$, showing that $\bar{f}$ is well defined.

Now $\bar{f}$ is a homomorphism, for $\bar{f}(aMbM) = \bar{f}(abM)$
$= f(ab)$

$= f(a)f(b) = \bar{f}(aM)\bar{f}(bM)$

Also $\bar{f}$ is one-to-one for if

$\bar{f}(aM) = \bar{f}(bM)$ then $f(a) = f(b)$

Here $f(a^{-1}b) = f(a^{-1}) \cdot f(b) = \{f(a)\}^{-1} f(b) = \{f(b)\}^{-1} f(b) = e'$

Therefore $a^{-1}b \in \ker f = M$.

Hence $b \in aM$. Therefore $bM = aM$

Thus $\bar{f}$ is one to one.

Finally since f is onto $\bar{f}$ is also onto. For let $a' \in G'$ be arbitrary. Then there exists an element $a \in G$ such that $f(a) = a'$.

Now $\bar{f}(aM) = f(a) = a'$, showing that a' is the image of $aM \in G/M$ under $\bar{f}$. Hence $\bar{f}$ is an onto homomorphism.

We have proved that $\bar{f}: G/M \to G'$ is an isomorphism

Hence $G/M \cong G'$. Also if $h: G \to G/M$ be the natural homomorphism of G onto G/M then for each $a \in G$.

Q State and Prove Cayley's theorem

Ans:- Statement:- Every group $(G, \cdot)$ is a isomorphic to permutation group on the set G.

Proof:- To each element $a \in G$, we associate the function f_a on the set G defined by $f_a(x) = ax$ for all $x \in G$. f_a is mapping of G into G.

Now f_a is one-to-one for

$f_a(x) = f_a(y) \Rightarrow ax = ay \Rightarrow x = y$

Also f_a is onto for given any $x \in G$, $a^{-1}x \in G$ is such that $f_a(a^{-1}x) = a(a^{-1}x) = (aa^{-1}x) = ex = x$. Thus f_a is a One-to-one map of G onto G. Hence f_a is a permutation on G even if G is infinite.

Now, let g be a map from the group G into the set of permutation on the set G defined by $g(a) = f_a$. We show that the set $H = \{f_a : a \in G\}$ is a group and that g is an isomorphism of G onto H. We first show that g is a homomorphism of G into the group of all permutation on the elements of G. For any $x \in G$, we have

$$\{g(ab)\}(x) = f_{ab}(x) = (ab)x$$

$$= a(bx) = f_a(bx) = f_a\{f_b(x)\} = f_a \cdot f_b(x)$$

Hence $g\{(ab)\} = f_a \cdot f_b = g(a) \cdot g(b)$. Hence g is a homomorphism and since $H =$ range of g, H is a group In order to show that g is an isomorphism of G onto H. Let $a \in \ker g$. Then $g(a) \cdot (x) = ex = x$ for all $x \in G$. Therefore $a = e$ and so g is an isomorphism.